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ADDENDUM TO "MULTIDOMAIN MULTIPHASE FLUID MECHANICS" [1]

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INTRODUCTION

Results of our continuing study affords further clarification of the basic interactions treated in the above publication. A significant finding is that the relations obtained from considering a dispersed multiphase system can be now derived from volume averaged dynamic equations, thus extending the ranges of consideration from multiphase to multidomain mixtures. The basic concepts of averaging have been extended from Birkhoff [2], Delhaye [3], and Slattery [4], via introducing conservation relations of the mixture. For conciseness, we shall only illustrate with the continuity and the momentum equations, leaving the general treatment for a future presentation.

PHASIC EQUATIONS

The phasic equations of continuum mechanics as applied to a pure phase are well known. In a pure phase k, the equations on continuity and momentum are:

$$\frac{\partial \rho_k}{\partial t} + \nabla\cdot(\rho_k \mathbf{U}_k) = 0 \tag{1}$$

$$\frac{\partial}{\partial t}(\rho_k U_k) + \nabla\cdot(\rho_k \mathbf{U}_k \mathbf{U}_k) = -\nabla P_k + \nabla\cdot\boldsymbol{\tau}_k + \rho_k \mathbf{f}_k \tag{2}$$

where t is the time, ρ_k is the density of pure phase k, $\mathbf{U}_k$ is its velocity, P_k is the static pressure inside phase k, $\mathbf{f}_k$ is the field force per unit mass, and $\boldsymbol{\tau}_k$ is the shear stress.

PHASE VOLUME AVERAGING

Within the framework of multiphase mechanics, bubbles (for instance) of different size ranges may constitute different phases [5]. Computationwise, we readily approximate a gradual change of sizes of particles by a gradual change in population of particles of discrete steps of particle mass.

Consider a region R of volume v of a mixture of phases enclosed by surface A, with R_k occupied by phase k over volume v_k, with A_{ke} denoting the intersection of A with A_k for phase k, and $\mathbf{n}_k$ is the outwardly directed unit normal to the phase interface $A_k - A_{ke} \equiv \Delta A_k$ from phase k. The volumetric averages of any scalar, vector, or tensor f associated with phase k are given by:

$$\tilde{f}_k = \frac{1}{v}\int_{R_k} f_k \,\mathrm{d}v \tag{3}$$

and

$$\langle f_k \rangle = \frac{1}{v_k}\int_{R_k} f_k \,\mathrm{d}v. \tag{4}$$

The volume averages of derivatives of this quantity are given from the generalized transport theorem as [4]:

$$\widetilde{\nabla f_k} \equiv \frac{1}{v}\int_{R_k} \nabla f_k \,\mathrm{d}v$$

$$= \nabla \tilde{f}_k + \frac{1}{v}\int_{A_k} f_k \mathbf{n}_k \,\mathrm{d}A \tag{5}$$

$$\widetilde{\nabla\cdot\mathbf{f}_k} = \nabla\cdot\tilde{\mathbf{f}}_k + \frac{1}{v}\int_{\Delta A_k} \mathbf{f}_k\cdot\mathbf{n}_k \,\mathrm{d}A \tag{6}$$

$$\frac{\widetilde{\partial f_k}}{\partial t} = \frac{\partial \tilde{f}_k}{\partial t} - \frac{1}{v}\int_{\Delta A_k} f_k \mathbf{U}_s\cdot\mathbf{n}_k \,\mathrm{d}A \tag{7}$$

where $\mathbf{U}_s\cdot\mathbf{n}_k$ is the speed of displacement of the phase interface. For $f_k = 1$ in equation (3), equation (7) gives:

$$\frac{\partial \alpha_k}{\partial t} = \frac{1}{v}\int_{\Delta A_k} \mathbf{U}_s\cdot\mathbf{n}_k \,\mathrm{d}A \tag{8}$$

and equation (5) gives:

$$\nabla \alpha_k = -\frac{1}{v}\int_{\Delta A_k} \mathbf{n}_k \,\mathrm{d}A. \tag{9}$$

Continuity equation

When these averaging relations are applied to specific quantities, we have the following relations:

$$\tilde{\rho}_k = \frac{1}{v}\int_{R_k} \bar{\rho}_k \,\mathrm{d}v = \alpha_k \bar{\rho}_k$$

$$\langle \rho_k \rangle = \frac{1}{v_k}\int_{R_k} \rho_k \,\mathrm{d}v = \bar{\rho}_k.$$

The latter for uniform density of material constituting phase k. Further, the densities in equations (1) and (2) are related to v_k where the density of phase k is the material density, or $\rho_k = \bar{\rho}_k$. When applied to the mixture over volume v, the average density becomes the density of the phase averaged over that volume, or $\rho_k = \tilde{\rho}_k$. This is a point that has to be understood in carrying out the averaging. $\tilde{\rho}_k$ is the density of a dispersed particle cloud although the average density of a phase in a system of stratified flow of two immiscible layers is less meaningful than in a dispersed system.

The momentum of phase k in a given volume in space is:

$$\widetilde{\rho_k \mathbf{U}_k} = \frac{1}{v}\int_{R_k} \rho_k \mathbf{U}_k \,\mathrm{d}v. \tag{10}$$

The averaging of the derivatives over volume v is carried out similarly, or when $\partial\bar{\rho}_k/\partial t$ in v_k is averaged over v:

$$\frac{1}{v}\int_{R_k} \frac{\partial \rho_k}{\partial t}\,\mathrm{d}v \equiv \frac{\widetilde{\partial \rho_k}}{\partial t} = \frac{\partial \tilde{\rho}_k}{\partial t} - \frac{1}{v}\int_{\Delta A_k} \rho_k \mathbf{U}_s\cdot\mathbf{n}_k \,\mathrm{d}A \tag{11}$$

where the first ρ_k from equation (1) is the material density in v_k and is now averaged over v, $\widetilde{\rho_k}$ is the density of the particle cloud, and the fourth ρ_k is the density of material constituting k inside k. Similarly, from equation (1):

$$\frac{1}{v}\int_{R_k} \nabla\cdot\rho_k\mathbf{U}_k\,\mathrm{d}v = \widetilde{\nabla\cdot\rho_k\mathbf{U}_k}$$

$$= \nabla\cdot\widetilde{\rho_k\mathbf{U}_k} + \frac{1}{v}\int_{\Delta A_k} \rho_k\mathbf{U}_k\cdot\mathbf{n}_k\,\mathrm{d}A \qquad (12)$$

where the first and the fourth $\rho_k\mathbf{U}_k$ are from equation (1) for material inside volume v_k, the third $\rho_k\mathbf{U}_k$ is now averaged over volume v. Therefore, from volume averaging of equation (1), we get:

$$\frac{\partial\tilde{\rho}_k}{\partial t} + \nabla\cdot(\widetilde{\rho_k\mathbf{U}_k}) + \frac{1}{v}\int_{\Delta A_k} \rho_k(\mathbf{U}_k - \mathbf{U}_s)\cdot\mathbf{n}_k\,\mathrm{d}A = 0 \quad (13)$$

or in terms of average density of phase k in volume v and the mean velocity $\tilde{\mathbf{U}}_k$:

$$\frac{\partial\tilde{\rho}_k}{\partial t} + \nabla\cdot(\tilde{\rho}_k\tilde{\mathbf{U}}_k) = \Gamma_k + \nabla\cdot\boldsymbol{\delta}_k \qquad (14)$$

where Γ_k is the rate of generation of k per unit volume v via conversion from the adjoining phase, and

$$\delta_k = \tilde{\rho}_k\tilde{\mathbf{U}}_k - \widetilde{\rho_k\mathbf{U}_k} \qquad (15)$$

which is zero for $\tilde{\rho}_k = \text{constant} = \rho_k$ (or for a single phase incompressible fluid, or for a two-phase mixture of incompressible fluids and constant α_k). δ_k remains to be determined. We note that $\Gamma_k = 0$ when the integral in equation (13) is zero. This interpretation differs from that in [3, 4, 6] and is based on the consideration of the general case of an interface of non-zero thickness [6].

Momentum equation

The average of pressure gradient in phase k is given by:

$$\widetilde{\nabla}P_k = \nabla\tilde{P}_k + \frac{1}{v}\int_{\Delta A_k} P_k\mathbf{n}_k\,\mathrm{d}A \qquad (16)$$

where P_k is the pressure at a point inside phase k, and

$$\tilde{P}_k = \frac{1}{v}\int_{R_k} P_k\,\mathrm{d}v = \alpha_k\langle P_k\rangle \qquad (17)$$

while

$$\langle P_k\rangle = \frac{1}{v_k}\int_{R_k} P_k\,\mathrm{d}v \qquad (18)$$

where $\langle P_k\rangle$ is the average pressure in the volume v_k. The integral in equation (16) is a pressure force due to interfacial pressure which works against $\nabla\tilde{P}_k$.

The meaning of the volume average of the shear stress is:

$$\widetilde{\nabla\cdot\boldsymbol{\tau}_k} = \nabla\cdot\tilde{\boldsymbol{\tau}}_k + \frac{1}{v}\int_{\Delta A_k} \boldsymbol{\tau}_k\cdot\mathbf{n}_k\,\mathrm{d}A \qquad (19)$$

where $\tilde{\boldsymbol{\tau}}_k$ is the shear stress of phase k in the mixture and the integral gives the force exerted by k on the boundary or drag force per unit volume.

Averaging equation (2) for phase k over volume v gives:

$$\begin{aligned}\frac{\partial}{\partial t}(\widetilde{\rho_k\mathbf{U}_k}) &- \frac{1}{v}\int_{\Delta A_k} \rho_k\mathbf{U}_k\mathbf{U}_s\cdot\mathbf{n}_k\,\mathrm{d}A + \nabla\cdot(\widetilde{\rho_k\mathbf{U}_k\mathbf{U}_k}) \\ &+ \frac{1}{v}\int_{\Delta A_k} \rho_k\mathbf{U}_k\mathbf{U}_k\cdot n_k\,\mathrm{d}A + \nabla\tilde{P}_k + \frac{1}{v}\int_{\Delta A_k} P_k\mathbf{n}_k\,\mathrm{d}A \\ &- \nabla\cdot\tilde{\boldsymbol{\tau}}_k - \frac{1}{v}\int_{\Delta A_k} \boldsymbol{\tau}_k\cdot\mathbf{n}_k\,\mathrm{d}A - \widetilde{\rho_k f_k} = 0. \qquad (20)\end{aligned}$$

Substituting the mean quantities $\tilde{\rho}_k$, $\tilde{\mathbf{U}}_k$ and taking $\mathbf{f}_k$ as a constant, we obtain:

$$\begin{aligned}\frac{\partial}{\partial t}(\tilde{\rho}_k\tilde{\mathbf{U}}_k) &+ \nabla\cdot(\tilde{\rho}_k\tilde{\mathbf{U}}_k\tilde{\mathbf{U}}_k) \\ &= -\nabla\tilde{P}_k + \nabla\cdot\tilde{\tau}_k + \tilde{\rho}_k\mathbf{f}_k \\ &\quad - \frac{1}{v}\int_{\Delta A_k} \rho_k\mathbf{U}_k(\mathbf{U}_k - \mathbf{U}_s)\cdot\mathbf{n}_k\,\mathrm{d}A \\ &\quad + \frac{1}{v}\int_{\Delta A_k} (-P_k\mathbf{n}_k + \boldsymbol{\tau}_k\cdot\mathbf{n}_k)\mathrm{d}A \\ &\quad + \frac{\partial}{\partial t}(\tilde{\rho}_k\tilde{U}_k - \widetilde{\rho_k U_k}) + \nabla\cdot(\tilde{\rho}_k\tilde{\mathbf{U}}_k\tilde{\mathbf{U}}_k - \widetilde{\rho_k\mathbf{U}_k\mathbf{U}_k}). \qquad (21)\end{aligned}$$

In terms of physical phenomena, we note that

$$-\frac{1}{v}\int_{\Delta A_k} \rho_k\mathbf{U}_k(\mathbf{U}_k - \mathbf{U}_s)\cdot\mathbf{n}_k\,\mathrm{d}A \equiv \mathbf{I}_{ks} + \Gamma_k\mathbf{U}' \qquad (22)$$

is related to the transfer of momentum at the interface which is dependent on the configuration and dynamics of the system and the generation of k. $\mathbf{I}_{ks}$ and $\mathbf{U}'$ remain to be determined, because Γ_k, when generated, will not be at $\mathbf{U}_k$ of that phase. The force $\mathbf{I}_{ks}$ may arise even in an inviscid system when one phase is accelerated relative to the other phase. This differs again from [3, 4, 6] from consideration of the general case of transfer of inertia force across the interface without phase change. The configuration of the interface may range from a dispersed system to a pure stratified flow system in which case no momentum is transferred through the interface except that due to friction and $\mathbf{I}_{ks} = 0$. The term

$$-\frac{1}{v}\int_{\Delta A_k} P_k\mathbf{n}_k\,\mathrm{d}A \equiv P_{ks}\nabla\alpha_k \qquad (23)$$

as suggested by $\nabla\alpha_k$ given by equation (9), is the pressure force at the interface due to interfacial mean pressure [6] which may work against that due to Brownian diffusion, for example. The integral

$$-\frac{1}{v}\int_{\Delta A_k} \boldsymbol{\tau}_k\cdot\mathbf{n}_k\,\mathrm{d}A = \tilde{\rho}_k F_{ks}(\tilde{\mathbf{U}}_s - \tilde{\mathbf{U}}_k) + V'_{ks} \equiv V_{ks} + V'_{ks} \qquad (24)$$

is the force due to shear stress at the interface where F_{ks} is defined as the inverse relaxation time for momentum transfer to the interface and V'_{ks} is the force due to change in the flow field. The last two terms in equation (21) consist of δ_k in equation (15) and

$$\boldsymbol{\Delta}_k = \tilde{\rho}_k\tilde{\mathbf{U}}_k\tilde{\mathbf{U}}_k - \widetilde{\rho_k\mathbf{U}_k\mathbf{U}_k} \qquad (25)$$

which also remains to be determined.

We further note that the physical meaning of $\tilde{\mathrm{P}}_k = \alpha_k\tilde{P}$ is seen in that

$$\nabla\alpha_k\langle P_k\rangle = \alpha_k\nabla\langle P_k\rangle + \langle P_k\rangle\nabla\alpha_k$$

where the first term on the right hand side $\alpha_k\nabla\langle P_k\rangle$ is the force acting on a phase due to pressure gradient and $\langle P_k\rangle\nabla\alpha_k$ is the force acting on a phase due to diffusion. The latter is expected to be reduced by $P_{ks}\nabla\alpha_k$ in equation (23). $\tilde{\boldsymbol{\tau}}_k$ in the above represents the shear stress of component k in the mixture and is equal to $\boldsymbol{\tau}_k$ in equation (2) for a pure phase only in the limiting case of stratified flow. When k is a dispersed phase (or a continuum phase containing a dispersed phase), $\tilde{\boldsymbol{\tau}}_k$ represents in the general sense, transfer of momentum by particles via bodily motion due to diffusion. For the general representation, we define:

$$\tilde{\boldsymbol{\tau}}_k \equiv \boldsymbol{\tau}_{km} \qquad (26)$$

or shear stress of a phase k in the mixture [5].

Equation (21) now becomes:

$$\frac{\partial}{\partial t}(\tilde{\rho}_k\tilde{\mathbf{U}}_k) + \nabla\cdot(\tilde{\rho}_k\tilde{\mathbf{U}}_k\tilde{\mathbf{U}}_k)$$

$$= -\nabla\alpha_k\langle P_k\rangle + P_{ks}\nabla\alpha_k + \tilde{\boldsymbol{\tau}}_k + \tilde{\rho}_k\mathbf{f}_k + \mathbf{V}_{ks} + \mathbf{V}'_{ks} + \mathbf{I}_{ks} + \Gamma_k\mathbf{U}' + \frac{\partial}{\partial t}\delta_k + \nabla\cdot\boldsymbol{\Delta}_k \tag{27}$$

Note that many terms remain to be determined:

1. Some parameters need to be revised such that in the multiphase computations, interface variables $\mathbf{U}_s$, etc. be replaced by proper interacting parameters replacing F_{ks}, etc. This reduces the number of dependent variables in the computation by eliminating variables at the interface except when treating discrete domains.
2. The effects of momentum transfer due to inertia of phases and the interface pressure across an interface: $\mathbf{I}_{ks}$, and $(\langle P_k\rangle - P_{ks})\nabla\alpha_k$ remain to be determined (they are zero for pure stratified flow) in the next section.
3. The effects of generation Γ_k and the associated velocities $\mathbf{U}'$ need to be identified.
4. All the deviations between products of the averages to the averages of the products remain to be determined.

VOLUME AVERAGING AS APPLIED TO A MIXTURE

In order to determine the terms identified in the categories of items (1), (2), and (3) at the end of the previous section, we explore the conservation principles in relation to an overall multiphase mixture. This step was not taken in [3, 4, 6] but is deemed to be a necessary step for a complete account of the multiphase system as a whole.

Excluding any solid structures in the system for the present, and summing over the phases, we have, according to equations (3) and (4):

$$\langle\rho_m\rangle = \sum_k \tilde{\rho}_k = \tilde{\rho}_m \tag{28}$$

$$\langle\rho_m\mathbf{U}_m\rangle = \sum_k \widetilde{\rho_k\mathbf{U}_k} = \widetilde{\rho_m U_m} \tag{29}$$

$$\langle\rho_m\mathbf{U}_m\mathbf{U}_m\rangle = \sum_k[\widetilde{\rho_k\mathbf{U}_k\mathbf{U}_k} - \widetilde{\rho_k(\mathbf{U}_k-\mathbf{U}_m)(\mathbf{U}_k-\mathbf{U}_m)}]$$

$$= \widetilde{\rho_m\mathbf{U}_m\mathbf{U}_m} \tag{30}$$

$$\langle\rho_m\mathbf{f}_m\rangle = \sum_k \widetilde{\rho_m\mathbf{f}_m} = \widetilde{\rho_m f_m} \tag{31}$$

$$\langle P_m\rangle = \sum_k \tilde{P}_k \tag{32}$$

$$\langle\boldsymbol{\tau}_m\rangle = \sum_k \tilde{\boldsymbol{\tau}}_k = \tilde{\boldsymbol{\tau}}_m = \sum \tilde{\boldsymbol{\tau}}_{km}. \tag{33}$$

(The last equalities are according to the definitions of the shear stress of component k in the mixture.) Expressions in equation (30) include a concern for objectivity [7].

The continuity equation (13) is readily summed over all the phases:

$$\frac{\partial\tilde{\rho}_m}{\partial t} + \nabla\cdot\widetilde{\rho_m U_m} = 0 \tag{34}$$

since from conservation at the interface:

$$\sum_k\int_{\Delta A_k}\rho_k(\mathbf{U}_k - \mathbf{U}_s)\cdot\mathbf{n}_k\,\mathrm{d}A = 0 \tag{35}$$

when reduced to individual averages, we get:

$$\frac{\partial\tilde{\rho}_m}{\partial t} + \nabla\cdot\tilde{\rho}_m\tilde{\mathbf{U}}_m = \nabla\cdot(\tilde{\rho}_m\tilde{\mathbf{U}}_m - \widetilde{\rho_m\mathbf{U}_m}) = \nabla\cdot\boldsymbol{\delta}_m. \tag{36}$$

The terms of the momentum equation, equation (20), are also readily summed, in particular:

$$\tilde{P} = \frac{1}{v}\int_R P\,\mathrm{d}v = \langle P\rangle = \sum_k\frac{1}{v}\int_{R_k}P_k\,\mathrm{d}v$$

$$= \sum\tilde{P}_k = \sum\alpha_k\langle P_k\rangle \tag{37}$$

or $\langle P_k\rangle = \tilde{P}$, although a difference may arise due to surface tension and large acceleration. Note that P is the pressure at a point in the mixture, and

$$\sum[\nabla(\alpha_k\langle P_k\rangle) - P_{ks}\Delta\alpha_k] = \nabla\langle P\rangle = \nabla\tilde{P} \tag{38}$$

suggesting that [6, 8]

$$P_{ks}\nabla\alpha_k = \tilde{P}_s\nabla\alpha_k \equiv B_k\tilde{P}\nabla\alpha_k \tag{39}$$

where $\tilde{P}_s$ is the mean interface pressure and B_k is a displacement factor which is a function of configuration of phase k and dynamic variables of the system [8]

Summing over all the phases k, we get

$$\frac{\partial}{\partial t}(\widetilde{\rho_m\mathbf{U}_m}) + \frac{1}{v}\sum_k\int_{\Delta A_k}\rho_k\mathbf{U}_k(\mathbf{U}_k - \mathbf{U}_s)\cdot\mathbf{n}_k\,\mathrm{d}A$$

$$+ \nabla\cdot(\widetilde{\rho_m\mathbf{U}_m\mathbf{U}_m}) + \nabla\tilde{P} - \nabla\cdot\tilde{\boldsymbol{\tau}}_m$$

$$-\frac{1}{v}\sum_k\int_{\Delta A_k}(-P_k\mathbf{I} + \boldsymbol{\tau}_k)n_k\,\mathrm{d}A - \tilde{\rho}_m\mathbf{f}_m$$

$$-\nabla\cdot\sum_k\rho_k(\mathbf{U}_k - \mathbf{U}_m)(\mathbf{U}_k - \mathbf{U}_k) = 0 \tag{40}$$

for constant $\mathbf{f}_m = \mathbf{f}_k$. We note that $\sum_k \Gamma_k U' = 0$, and $U = \mathbf{U}_m$.

The jump condition momentum balance at the interface gives:

$$-\frac{1}{v}\sum_k\int_{\Delta A_k}[-\rho_k U_k(U_k - U_s) - P_k\mathbf{I} + \boldsymbol{\tau}_k]\cdot\mathbf{n}_k\,\mathrm{d}A$$

$$= \sum_k\mathbf{I}_{ks} \equiv \sum_k\mathbf{I}_k. \tag{41}$$

Therefore, we can rewrite:

$$\frac{\partial}{\partial t}(\tilde{\rho}_m\tilde{\mathbf{U}}_m) + \nabla\cdot(\tilde{\rho}_m\tilde{\mathbf{U}}_m\tilde{\mathbf{U}}_m) + \sum\mathbf{I}_k + \nabla\tilde{P} - \nabla\cdot\boldsymbol{\tau}_m$$

$$-\tilde{\rho}_m\mathbf{f}_m - \nabla\cdot\sum_k\tilde{\rho}_k(\tilde{\mathbf{U}}_k - \tilde{\mathbf{U}}_m)(\tilde{\mathbf{U}}_k - \tilde{\mathbf{U}}_m)$$

$$= \frac{\partial}{\partial t}\delta_m + \nabla\cdot\boldsymbol{\Delta}_m - \nabla\cdot\boldsymbol{\Delta}_{Im} \tag{42}$$

where

$$\delta_m = \tilde{\rho}_m\tilde{\mathbf{U}}_m - \widetilde{p_m\mathbf{U}_m} \tag{43}$$

$$\Delta_m = \tilde{\rho}_m\tilde{\mathbf{U}}_m\tilde{\mathbf{U}}^{\xi} - \widetilde{\rho_m\mathbf{U}_m\mathbf{U}_m} \tag{44}$$

$$\Delta_{Im} = \sum_k[\tilde{\rho}_k(\tilde{\mathbf{U}}_k - \tilde{\mathbf{U}}_m)(\tilde{\mathbf{U}}_k - \tilde{\mathbf{U}}_m)$$

$$- \widetilde{\rho_k(\mathbf{U}_k-\mathbf{U}_m)(\mathbf{U}_k-\mathbf{U}_m)}]$$

$$= \sum\boldsymbol{\Delta}_k - \sum(\tilde{\rho}_k\tilde{\mathbf{U}}_k\tilde{\mathbf{U}}_m - \widetilde{\rho_k\mathbf{U}_k\mathbf{U}_m}) + \boldsymbol{\Delta}_m. \tag{45}$$

We note that $\mathbf{I}_{ks}$ or $\mathbf{I}_k = 0$ for pure stratified flow where there is no transfer of inertia across the interface. For highly dispersed multiphase mixture [9], the nature of conservation based on the mixture is prominent, in which case:

$$I_k = \nabla\cdot\tilde{\rho}_k(\tilde{\mathbf{U}}_k - \tilde{\mathbf{U}}_m)(\tilde{\mathbf{U}}_k - \tilde{\mathbf{U}}_m). \tag{46}$$

In general, we may designate:

$$I_k = C_k\nabla\cdot\tilde{\rho}_k(\tilde{\mathbf{U}}_k - \tilde{\mathbf{U}}_m)(\tilde{\mathbf{U}}_k - \tilde{\mathbf{U}}_m) \tag{47}$$

where C_k is a dispersion factor which is a function of the configuration of phase k and dynamic variables of the system, (highly dispersed and locally isotropic, $C_k = 1$; or purely stratified and anisotropic, $C_k = 0$).

In between, situations such as distributed bubble chain, slug flow, wavy stratified flow, or annular flow, $1 \geq C_k \geq 0$. The displacement factor, B_k, in equation (39), however, has the range of $0 \leq B_k \leq 1$, with the limits of $B_k = 0$ for dispersed multiphase flow where diffusion is prominent and $B_k = 1$ for pure stratified flow (multidomain). A discussion of the influence of the whole range of flow configurations is given in a separate paper by Chao, Sha, and Soo [10]. While C_k and B_k cannot be determined rigorously at this stage except for the limiting cases, they suggest that the dynamic equations of a multiphase system have to account for the configuration of phases rather than being a simple carry over from the single phase flow system.

Corresponding relations for the energy equation are readily derived. The force due to virtual mass $\mathbf{I}_k'$ and its rate of doing work L_k' are readily included in as cumulative with $\mathbf{I}_k$ and L_k [1].

It might be said that even though the terms including the δs and the Δs still remain to be resolved, the present averaging effort has helped to extend the earlier formulation from multiphase to distributed multidomain (including distributed stratified flow). The multiphase formulation remains the same as before [1] when differences between the products of averages and averages of products and their derivatives are assumed to be zero.

If the formulations based on averaging are to be applied to the calculations of real systems, Δs, δs, and their derivatives must be determined. Tacit assumption of zero values for these quantities is tantamount to taking individual averaged parameters in the beginning.

In the general sense, the case of $C_k = 0$ and $B_k = 1$, or the drift flux model, is an idealization or limiting case. It is an adequate approximation in some cases when the effect of the passage of pressure and inertia waves across the interface does not have an important effect. However, when this abstraction is applied to an analysis of wave interaction such as in a one-dimensional transient flow analysis, it yields imaginary characteristics while the same analysis as applied to each phase gives real characteristics. The imaginary result indicates that, when transfer of waves across the interface is excluded, a common set of characteristics for the mixture does not exist. For the general case of $1 > C_k > 0$ and $0 < B_k < 1$, the characteristics are real in a multiphase-multidomain system [11].

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